Associative memory storing an extensive number of patterns based on a network of oscillators with distributed natural frequencies in the presence of external white noise

Masahiko Yoshioka* and Masatoshi Shiino†
Department of Applied Physics, Tokyo Institute of Technology Ohokayama, Meguro-ku, Tokyo 152, Japan
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We study associative memory based on temporal coding in which successful retrieval is realized as an entrainment in a network of simple phase oscillators with distributed natural frequencies under the influence of white noise. The memory patterns are assumed to be given by uniformly distributed random numbers on $[0, 2\pi)$ so that the patterns encode the phase differences of the oscillators. To derive the macroscopic order parameter equations for the network with an extensive number of stored patterns, we introduce an effective transfer function by assuming a fixed-point equation of the form of the Thouless-Anderson-Palmer equation, which describes the time-averaged output as a function of the effective time-averaged local field. Properties of the networks associated with synchronization phenomena for a discrete symmetric natural frequency distribution with three frequency components are studied based on the order parameter equations, and are shown to be in good agreement with the results of numerical simulations. Two types of retrieval states are found to occur with respect to the degree of synchronization, when the size of the width of the natural frequency distribution is changed.

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I. INTRODUCTION

Recently a population of neurons in the cat visual cortex has been reported to exhibit synchronized firings in a stimulus dependent manner [1,2]. The occurrence of correlations in firing times of neurons seems to be a ubiquitous phenomenon in real nervous systems. The role of such synchronized firings for information processing in the brain has been attracting growing interest of researchers, and several authors have suggested neural network models based on the concept of temporal coding, where information of a neuron is represented by its firing times. Indeed, to explain the experimental findings that visual information of an external object is processed with being divided into several pieces of information, several authors [3–6] have suggested that the synchronized firings of neurons may serve as a linker for those pieces of information.

The problem of investigating how an associative memory is realized in real nervous systems as well as of constructing biologically relevant models is of central concern of neuroscientists. Since the establishment of systematic theories of associative memory for networks with an energy function that is ensured by assuming symmetric synaptic couplings [7–16], several attempts have been made to make models as biologically plausible as possible [17–19]. Previously we investigated the effects of asymmetric couplings [20] for memorizing presynaptic and post-synaptic activities, which are incorporated into the standard symmetric Hebb learning rule [21], by studying networks of analog neurons [22,23], whose continuous-time dynamics involves a positive-valued transfer function representing the mean firing rates of a neuron as a function of membrane potential.

Instead of working with the concept of rate coding based on the idea that neuronal information is represented by mean firing rate of a single neuron, one may be concerned with the concept of temporal coding, when considering that spatiotemporal patterns of neuronal firings will make information carried by a population of neurons much richer than spatial patterns alone. A spiking neuron is considered to be one of the candidates for implementation of temporal coding [24–33]. The time evolution of membrane potential that is generated in response to an injected synaptic electric current of a spiking neuron is described by such nonlinear dynamics as Hodgkin-Huxley equation [34], FitzHugh-Nagumo equation [35,36], or the equation of an integrated-and-fire neuron. Spiking neurons in a network are generally supposed to interact with each other via pulses generated in the firing events occurring in the pre-synaptic neuron. We have shown previously that even in the presence of time delays in transmission of the pulses an associative memory based on a network of spiking neurons can be realized by assuming a simple Hebb-type learning rule alone, and that the memory retrieval accompanies synchronized firings of neurons. The dynamics of such associative memory was analyzed by means of sublattice method in our previous paper [37].

In the previous analysis [37] we assumed that every neuron shares identical characteristics to exhibit the same reaction in response to the same injected current. In real nervous systems, however, neurons in a network may have their own individual characteristics.

The problem of whether temporal coding functions robustly in the presence of certain heterogeneities of neurons will be of particular interest. For the purpose of investigating such a problem for the phenomenon of synchronized oscillations in associative memory neural networks, we consider it appropriate to deal with simple models of coupled oscillators with distributed natural frequencies and external noise. It is well known that, under certain conditions, a population of oscillators with distributed natural frequencies is allowed to

*Electronic address: myosioka@mikau.ap.titech.ac.jp
†Electronic address: mshiino@ap.titech.ac.jp

get partially entrained in such a way that oscillators with a natural frequency near the central frequency become to oscillate synchronously at identical frequency as a result of cooperative interactions [38–40].

Kuramoto [38] showed that the dynamics of this kind of network of oscillators with sufficiently weak interactions can be reduced to a simple phase dynamics. Supposing that neurons in a network are treated as phase oscillators, associative memory has been shown to be realized under a simple learning rule of the Hebb-type either in the case of a finite number of stored patterns [41,39] or in the case with a single natural frequency [42,43]. Satisfactory analysis of the case with a distribution of natural frequency and extensively many stored patterns has been far less conducted. Quite recently, we have reported the study of a deterministic phase oscillator network with a distribution of natural frequencies where an extensive number of binary patterns (± 1) are stored with use of the Hebb learning rule [44].

The main purpose of the present study is the theoretical analysis of associative memory based on temporal coding with use of networks of phase oscillators in the more general case where the number of stored patterns that are given by uniformly distributed random numbers on $[0, 2\pi)$ is extensive, natural frequencies of the oscillators are distributed according to a certain distribution function, and furthermore external white noise is added to the system.

While one can analyze a phase oscillator network with a single natural frequency by means of the replica method that makes full use of its associated energy function, one cannot resort any more to the standard method of statistical physics based on the existence of an energy function in the case of networks with a distribution of natural frequencies.

One can, however, use the self-consistent signal-to-noise analysis (SCSNA) [45,46,22,23,37] to deal with general cases without energy functions. To apply the SCSNA it is necessary to know fixed-point equations describing the equilibrium states of the network. When considering such equations in stochastic systems, we may take advantage of the concept of the Thouless-Anderson-Palmer (TAP) equation [47,48].

The Thouless-Anderson-Palmer (TAP) equation is known to exist for the Sherington-Kirkpatrick (SK) model of spin glasses [48–50] and the Hopfield model of an Ising spin neural networks [8,48,51], and to represent a functional relationship between the thermal or time average of each spin in equilibrium and its corresponding effective local field that involves the so-called Onsager reaction term [48]. Usefulness of the TAP equation in deriving the order parameter equations of associative memory networks is attributed to the fact that the resulting equations of the replica calculations by Amit *et al.* is recovered by the result of application of the SCSNA to the TAP equation [46], where the Onsager reaction term is canceled exactly by the appearance of the renormalized output term of the SCSNA [46]. We note that the TAP equation of the naive mean field model with the interactions given by the Hebb learning rule defines to an analog network equation with the transfer function tanh (βh) [46].

We first evaluate an analogue of the TAP equation of the naive mean-field type for our model with a distribution of natural frequencies by dealing with the Fokker-Planck equation. In order to obtain the Onsager reaction term we compute the free energy of the network without a distribution of natural frequencies to derive the TAP equation by following the method of Plefka [50] and Nakanishi [51]. Then we assume that the TAP-like equation also exists with the Onsager reaction term remaining the same even for networks with a distribution of natural frequencies, and that such a TAP-like equation defines an effective transfer function to which the SCSNA is applicable to obtain the order parameter equations.

The present paper is organized as follows. In Sec. II, we introduce a neural network of simple phase oscillators and describe how the network functions as an associative memory based on a simple learning rule of the Hebb-type. In Sec. III, we give a theoretical analysis to derive the macroscopic order parameter equations describing the long time behavior of the system. On the basis of the order parameter equations, in Sec. IV we investigate properties of memory retrieval accompanying synchronization in the networks by assuming a discrete symmetric natural frequency distribution with three frequency components. Results of numerical simulations are presented showing good agreement with those of theoretical analysis. In Sec. V, comparing our work with those of other researchers conducted previously, we summarize the results of the present study.

II. NEURAL NETWORKS OF PHASE OSCILLATORS WITH DISTRIBUTED NATURAL FREQUENCIES

The system under consideration is a network of N phase oscillators subjected to external white noise, whose dynamics is expressed as

$$\dot{\phi}_i = \omega_i - \sum_{j \neq i}^{N} J_{ij} \sin(\phi_i - \phi_j - \beta_{ij}) + \eta_i(t), \tag{1}$$

where ϕ_i and ω_i represent the phase and the natural frequency of oscillator i, respectively. β_{ij} and J_{ij} represent a certain phase shift and the strength of coupling between oscillator i and j, respectively. The Gaussian white noise $\eta_i(t)$ is assumed to satisfy $\langle \eta_i(t) \rangle = 0$ and $\langle \eta_i(t) \eta_j(t') \rangle = 2D \delta_{ij} \delta(t-t')$.

When $\omega_i = 0 (i = 1, \ldots, N)$, $c_{ij} = J_{ij} \exp(i\beta_{ij})$ satisfy $c_{ij} = c_{ji}^{*}$ with $*$ denoting complex conjugation and J_{ij} take real values, the system (1) has the energy function:

$$H(\{\phi_i\}) = -\frac{1}{2} \sum_{i \neq j} J_{ij} \cos(\phi_i - \phi_j - \beta_{ij}). \tag{2}$$

Then one has an equilibrium probability distribution $\rho(\{\phi_i\})$ proportional to $\exp[-H(\{\phi_i\})/D]$. In the case of $D = 0$, the function (2) becomes a Lyapunov function of the system and hence the state of the system eventually settles into a certain fixed point attractor after a long time.

In the present study, we assume natural frequencies to be distributed accordingly to an even distribution function $g(\omega) = g(-\omega)$ so that the average of natural frequencies become zero without loss of generality. To store P quenched random patterns $\theta_i^{\mu} (i = 1, \ldots, N, \mu = 1, \ldots, P)$ chosen from the uniform distribution on the interval $[0, 2\pi)$, we assume the Hebb type learning rule, and set the parameter β_{ij} and real valued J_{ij} such that

$$c_{ij}=J_{ij}\exp(i\beta_{ij})=\begin{cases}\frac{1}{N}\sum_{\mu=1}^{P}\xi_i^\mu\xi_j^{\mu *} & i\neq j\\ 0 & i=j\end{cases}, \tag{3}$$

where $\xi_i^\mu=\exp(i\theta_i^\mu)$. This definition of couplings gives networks the following properties:

(1) In successful retrieval an entrainment occurs, where synchronized oscillators satisfy $\phi_i-\phi_j\approx\theta_i^\mu-\theta_j^\mu$ with a target pattern μ recalled. (Note that if ϕ_i is the solution of dynamics (1), uniformly shifted phase $\tilde{\phi}_i=\phi_i+c$ also becomes its solution. What matters is not the phase itself but their difference $\phi_i-\phi_j$.)

(2) In the case of unsuccessful retrieval, all the oscillators fail to synchronize, running at their own natural frequencies.

To measure the distance between the pattern μ and the state of the system, we introduce the overlap for pattern μ

$$m^\mu(t)=\frac{1}{N}\sum_i \xi_i^{\mu *}z_i(t), \tag{4}$$

where we denote $e^{i\phi(t)}$ by $z_i(t)$. Then by use of the local field

$$h_i(t)=\sum_{j\neq i}c_{ij}z_j(t)=\sum_\mu \xi_i^\mu m^\mu(t)-\alpha z_i, \tag{5}$$

the dynamics (1) is rewritten as

$$\dot{\phi}_i=\omega_i-\mathrm{Re}\{h_i(t)\}\sin\phi_i+\mathrm{Im}\{h_i(t)\}\cos\phi_i+\eta_i(t), \tag{6}$$

where α denotes the loading rate P/N. From Eq. (6) it is easy to see that the learning rule (3) indeed realizes the above mentioned properties if the number of stored patterns is finite ($\alpha=0$) and $\omega_i=0$ $(i=1,\ldots,N)$. The memory retrieval accompanying synchronization can also occur for $\alpha>0$ even in the presence of a distribution of natural frequencies.

III. MACROSCOPIC ORDER PARAMETER EQUATIONS

Behaviors of associative memory networks depend crucially on the nature of the local fields or the neurons of oscillators, because the updating rule for the time evolution of the system is based on the local fields as is seen in Eq. (6). When the long time behavior of a network is described by fixed point type attractors, the relation between the resulting output state of a neuron and the corresponding local fields becomes essential for determining equilibrium properties of the networks. Such a relation is naturally introduced in the case of deterministic analog networks where neurons are characterized by transfer functions describing the input-output relation.

For such stochastic systems (1) as Ising spin networks, equilibrium fixed-point equations called the TAP equations are known to exist as expressing the relation between time average of each output of a neuron and its corresponding effective local fields involving the so-called Onsager reaction term that is proportional to the time-averaged output. Once the TAP-like equation level description is available, the SCSNA, in which one computes the variance of the cross-talk noises in the local field as a result of storing an extensive

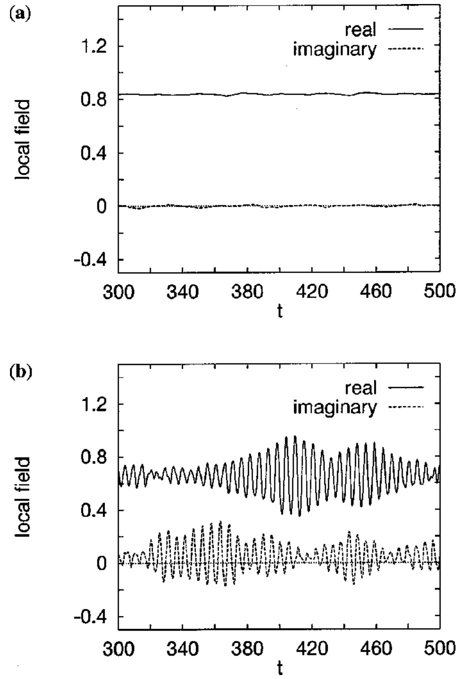

FIG. 1. The typical time evolution of the local fields observed in numerical simulations with $N=8000$, $\alpha=0.02$, $D=0$. Natural frequencies are chosen so as to obey (a) a Gaussian distribution $g(\omega)=\exp(-\omega^2/2\sigma^2)/\sqrt{2\pi\sigma^2}$ with $\sigma=0.3$ and (b) $g(\omega)=0.15\delta(\omega+1.4)+0.7\delta(\omega)+0.15\delta(\omega-1.4)$.

number of patterns, can be applied to obtain the order parameter equations in the limit $N\to\infty$.

A. Effective transfer function based on time-averaged local field and the TAP-like equation

If $\omega_i=0$ $(i=1,\ldots,N)$ and $D=0$, the state of the network eventually settles into an equilibrium state given by a fixed-point attractor owing to the existence of the Lyapunov function (2), and then the local fields do not fluctuate in time. Even in the presence of external white noise $(D\neq 0)$, the local fields also get fixed in time after a long time, provided $\alpha=P/N=0$. When the local fields are fixed over the time change due to the law of large numbers, theoretical treatment becomes simple because one can reduce the many body problem to a single-body problem.

In the more general case where $\alpha>0$ and/or natural frequencies are distributed, the local fields may fluctuate even with a large number of oscillators as can be shown in the numerical simulation illustrated in Fig. 1. The fluctuations seem to be aperiodic and rigorous analysis of such fluctuations is quite difficult. To deal with this situation we are forced to resort to a certain approximation by confining ourselves only to the near equilibrium behavior of the system: *we replace the time-dependent local fields by their time-averaged ones*

$$h_i(t)\approx\overline{h_i}=\sum_{j\neq i}c_{ij}\overline{z_j}, \tag{7}$$

where the overbar represents the time average at near equilibrium.

Once we apply this approximation, we are allowed to treat each interconnected oscillator as an element obeying the dynamics of one degree

$$\dot{\phi}_i = \omega_i - \mathrm{Re}(\overline{h_i})\sin\phi_i + \mathrm{Im}(\overline{h_i})\cos\phi_i + \eta_i(t), \tag{8}$$

which is expected to describe the behavior of the oscillator i near equilibrium. It turns out that the time average $\overline{z_i}$ at equilibrium can be expressed as a function of ω_i and $\overline{h_i}$: $\overline{z_i} = f(\omega_i, \overline{h_i}, \overline{h_i^*})$. We call it the transfer function in the case of network of analog neurons. It is easy to show that the transfer function $f(\omega, \overline{h}, \overline{h^*})$ satisfies

$$f[\omega, re^{i\theta}, (re^{i\theta})^*] = e^{i\theta} f(\omega, r, r). \tag{9}$$

Hence it suffices to calculate $f(\omega, r, r)$ to obtain $f[\omega, re^{i\theta}, (\mathrm{re}^{i\theta})^*]$. In the absence of white noise $(D=0)$, we can easily obtain the transfer function for real-valued local field r [38]:

$$f(\omega,r,r) = \begin{cases} \dfrac{i}{r}(\omega + \sqrt{\omega^2 - r^2}), & \omega < -r \\ \dfrac{1}{r}(i\omega + \sqrt{r^2 - \omega^2}), & -r < \omega < r \\ \dfrac{i}{r}(\omega - \sqrt{\omega^2 - r^2}), & r < \omega. \end{cases} \tag{10}$$

In deriving Eq. (10) in the case of $-r < \omega < r$ we used $\dot{\phi} = 0$ together with $d\dot{\phi}/d\phi < 0$. In the case of $\omega < -r$ or $r < \omega$ we computed the time average of $\exp[i\phi(t)]$ over the period T_0 of the periodic oscillations of ϕ:

$$f(\omega,r,r) = \frac{1}{T_0}\int_0^{T_0} \exp[i\phi(t)]dt = \frac{\displaystyle\int_0^{2\pi} \frac{e^{i\phi}}{\dot{\phi}} d\phi}{\displaystyle\int_0^{2\pi} \frac{1}{\dot{\phi}} d\phi}$$

$$= \frac{\displaystyle\int_0^{2\pi} \frac{e^{i\phi}}{\omega - r\sin\phi} d\phi}{\displaystyle\int_0^{2\pi} \frac{1}{\omega - r\sin\phi} d\phi}. \tag{11}$$

In Fig. 2, we illustrate the shape of the transfer function $f(\omega, \overline{h}, \overline{h^*})$ that is obtained from Eqs. (9) and (10).

In the presence of white noise $(D>0)$, from the Langevin equation (8) we obtain the Fokker-Planck equation

$$\dot{\rho}(\phi, t; \omega, \overline{h}) = -\frac{\partial}{\partial\phi}\{[\omega - \mathrm{Re}(\overline{h})\sin\phi + \mathrm{Im}(\overline{h})\cos\phi]\rho\} + D\frac{\partial^2\rho}{\partial\phi^2}, \tag{12}$$

where $\rho(\phi, t; \omega, \overline{h})$ is the probability distribution of phase $\phi \in [0, 2\pi]$ at time t, and periodic boundary conditions $\rho(0, t; \omega, \overline{h}) = \rho(2\pi, t; \omega, \overline{h})$, $d\rho(0, t; \omega, \overline{h})/d\phi = d\rho(2\pi, t; \omega, \overline{h})/d\phi$ are imposed. Since we are concerned with the probability distribution $\rho_{eq}(\phi; \omega, r)$ attained after a long time, we set $\dot{\rho}(\phi, t; \omega, r) = 0$ to obtain [39,41]

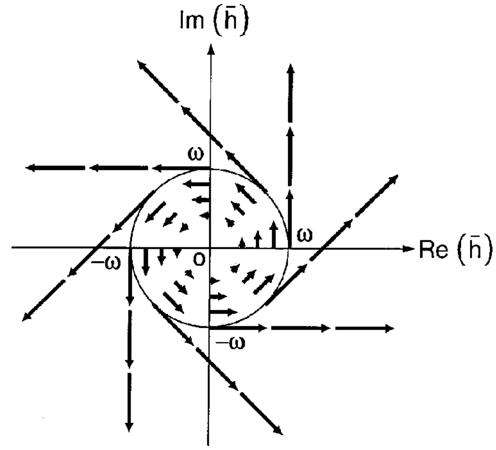

FIG. 2. Graphical representation of the effective transfer function $f(\omega, \overline{h}, \overline{h^*})$ in the case with $D=0$ and $\omega>0$. The output $f(\omega, \overline{h}, \overline{h^*})$ is represented by a vector at the position $\overline{h}$ on the complex plane. In the region, where $|\overline{h}|<1$, oscillators get synchronized with $|f(\omega, \overline{h}, \overline{h^*})| = 1$, while inside the circle oscillators get desynchronized with $|f(\omega, \overline{h}, \overline{h^*})| < 1$. In the case of $\omega<0$ the rotational direction of the flow pattern gets reversed owing to the property $f(-\omega, \overline{h}, \overline{h^*}) = \{f(\omega, \overline{h^*}, \overline{h})\}^*$.

$$\rho_{eq}(\phi; \omega, r) = \frac{I(\phi)}{\displaystyle\int_0^{2\pi} I(\phi) d\phi} \tag{13}$$

with

$$I(\phi) = \exp(\tilde{r}\cos\phi)\int_0^{2\pi} \exp\{-\tilde{\omega}\varphi - \tilde{r}\cos(\varphi + \phi)\} d\varphi, \tag{14}$$

where $\tilde{\omega} = \omega/D$ and $\tilde{r} = r/D$. Noting the ergodic property on the Fokker-Planck equation (12), which holds when $\overline{h}$ is viewed as a given parameter, we obtain the time average of z by computing the average over the equilibrium distribution $\rho_{\mathrm{eq}}(\phi; \omega, r)$

$$f(\omega, r, r) = \frac{\displaystyle\int_0^{2\pi} e^{i\phi} I(\phi) d\phi}{\displaystyle\int_0^{2\pi} I(\phi) d\phi}. \tag{15}$$

The transfer function we have obtained here can be considered to be an analogue of the so called TAP equation by the naive mean field model, because the time average of the output $\overline{z}$ is represented as a function of time-averaged local fields $\overline{h}$. We note, however, that the genuine TAP equation, which is defined for systems with an energy function, should describe the functional relation between the time averaged output and the effective local field that differs in general from the time-averaged one. The difference between the two types of local fields is known to be the Onsager reaction term in the theory of random spin systems. We assume that the TAP-like equation may hold even for the present system without an energy function, and suppose it to be given by the following equations

$$\overline{z_i}=f(\omega_i,h_i^{\rm TAP},h_i^{\rm TAP*}), \tag{16}$$

$$h_i^{\rm TAP}=\overline{h_i}+\gamma^{\rm TAP}\overline{z_i}$$
$$=\sum_{j\neq i}c_{ij}\overline{z_j}+\gamma^{\rm TAP}\overline{z_i}=\sum_{\mu}\xi_i^{\mu}\overline{m^{\mu}}-\alpha\overline{z_i}+\gamma^{\rm TAP}\overline{z_i}. \tag{17}$$

In the case with $\omega_i=0$ $(i=1,\ldots,N)$, by evaluating the free energy of the system, we can derive an explicit expression of the coefficient $\gamma^{\rm TAP}$ taking the form (see Appendix A)

$$\gamma^{\rm TAP}=-\alpha\frac{(1-q)/2D}{1-(1-q)/2D}, \tag{18}$$

where

$$q=\frac{1}{N}\sum_i\left|\overline{z_i}\right|^2. \tag{19}$$

It will, however, be difficult to rigorously derive an expression of $\gamma^{\rm TAP}$ for the general case with a natural frequency distribution. Thus we are led to make an assumption that the legitimate expression (18) for the case with $\omega_i=0$ $(i=1,\ldots,N)$ can naturally be extended to the general case. Describing the desired $\gamma^{\rm TAP}$ requires the introduction of the order parameter u that appears in the SCSNA.

B. Self-consistent signal-to-noise analysis

We consider the case with $m^1=\mathcal{O}(1)$ and $m^\mu=\mathcal{O}(1/\sqrt{N})$ $(\mu=2,\ldots,P)$, where we choose pattern 1 as the target. Assuming, without loss of generality, that $\xi_i^1=1$ for all i and m^1 is real owing to the rotational symmetry, the local field Eq. (17) is rewritten in the form:

$$h_i=m^1+\frac{1}{N}\sum_{\mu>1}\sum_{j\neq i}\xi_i^\mu\xi_j^{\mu*}s_j+\gamma^{\rm TAP}s_i, \tag{20}$$

where we have used s_i, h_i, and m^μ to represent respectively $\overline{z_i}$, $\overline{h_i}$, and $\overline{m^\mu}$ for brevity.

When we consider the case with a finite number of stored patterns, the analysis is straightforward since we already know the form of the transfer functions (10) $(D=0)$ and (15) $(D\neq 0)$. Since $\gamma^{\rm TAP}=0$ and $h_i=m^1$ in this case, we have in the limit $N\to\infty$

$$m^1=\frac{1}{N}\sum_i\xi_i^{1*}s_i\to\langle f(\omega,m^1,m^1)\rangle_\omega, \tag{21}$$

where $\langle\cdots\rangle_\omega=\int g(\omega)\cdots d\omega$. Solving this equation numerically, we evaluate the size of the overlap as a function of various parameters including D.

In the case of an extensive number of stored patterns $(\alpha>0)$, however, the cross-talk noise [the second term of Eq. (20)] in the local fields becomes to an appreciable extent, then one has to employ the method of SCSNA. The crux of the SCSNA is the evaluation of the variance of cross-talk noise that interferes occurrence of retrieval state.

According to the prescription of the SCSNA [23], the local field (20) is assumed to be in the form:

$$h_i=m^1+\frac{\lambda}{N}\sum_{\mu>1}\sum_{j\neq i}\xi_i^\mu\xi_j^{\mu*}s_j^\mu+\gamma^{\rm SCSNA}s_i+\gamma^{\rm TAP}s_i, \tag{22}$$

where s_j^μ, whose explicit expression is given later, is a quantity that is very close to s_j^μ and has a negligible correlation in the limit $N\to\infty$, and λ and $\gamma^{\rm SCSNA}$ are to be self-consistently determined in the course of analysis.

Defining $\tilde{h}_i=m^1+(\lambda/N)\Sigma_{\mu>1}\Sigma_{j\neq i}\xi_i^\mu\xi_j^{\mu*}s_j^\mu$, Eq. (16) reads

$$s_i=f[\omega_i,\tilde{h}_i+\gamma^{\rm TOTAL}s_i,(\tilde{h}_i+\gamma^{\rm TOTAL}s_i)^*], \tag{23}$$

where $\gamma^{\rm TOTAL}=\gamma^{\rm SCSNA}+\gamma^{\rm TAP}$.

Considering a general case with $\gamma^{\rm TOTAL}\neq 0$, we solve Eq. (23) for s_i, to obtain the renormalized output $s_i=\tilde{f}(\omega_i,\tilde{h}_i,\tilde{h}_i^*)$. Performing a Taylor expansion of $\tilde{f}(\omega_i,\tilde{h}_i,\tilde{h}_i^*)$ about $(\tilde{h}_i^\mu,\tilde{h}_i^{\mu*})$, we have

$$s_i=s_i^\mu+\left.\frac{\partial\tilde{f}}{\partial\tilde{h}}\right|_{(\tilde{h}_i^\mu,\tilde{h}_i^{\mu*})}\frac{\lambda}{N}\sum_{j\neq i}\xi_i^\mu\xi_j^{\mu*}s_j^\mu+\frac{\partial\tilde{f}}{\partial\tilde{h}^*}_{(\tilde{h}_i^\mu,\tilde{h}_i^{\mu*})}\times\left(\frac{\lambda}{N}\sum_{j\neq i}\xi_i^\mu\xi_j^{\mu*}s_j^\mu\right)^* \tag{24}$$

with

$$\tilde{h}_i^\mu=m^1+\frac{\lambda}{N}\sum_{\nu\neq 1,\mu}\sum_{j\neq i}\xi_i^\nu\xi_j^{\nu*}s_j^\nu, \tag{25}$$

$$s_i^\mu=\tilde{f}(\omega_i,\tilde{h}_i^\mu,\tilde{h}_i^{\mu*}). \tag{26}$$

Substituting Eq. (24) into Eq. (20) and comparing the result with Eq. (22) (see Appendix B for details) we obtain

$$m^1+\frac{\lambda}{N}\sum_{\mu>1}\sum_{j\neq i}\xi_i^\mu\xi_j^{\mu*}s_j^\mu+\gamma^{\rm SCSNA}s_i+\gamma^{\rm TAP}s_i$$
$$=m^1+\frac{1+u\lambda}{N}\sum_{\mu>1}\sum_{j\neq i}\xi_i^\mu\xi_j^{\mu*}s_j^\mu+\alpha u\lambda s_i+\gamma^{\rm TAP}s_i \tag{27}$$

where

$$u=\frac{1}{N}\sum_i\left.\frac{\partial\tilde{f}}{\partial\tilde{h}}\right|_{(\tilde{h}_i,\tilde{h}_i^*)}. \tag{28}$$

Since this equation holds for every site i, λ and $\gamma^{\rm SCSNA}$ are self-consistently determined as

$$\lambda=\frac{1}{1-u}, \tag{29}$$

$$\gamma^{\text{SCSNA}} = \alpha \frac{u}{1-u}. \tag{30}$$

The variance of renormalized noise $\tilde{N}_i = (\lambda/N)\Sigma_{\mu>1}\Sigma_{j\neq i}\xi_i^\mu \xi_j^{\mu *} s_j^\mu$ in the right-hand side of Eq. (22) can be evaluated by noting that $\text{Re}(\tilde{N}_i)$ and $\text{Im}(\tilde{N}_i)$ distribute over sites obeying an identical Gaussian distribution independently and the site average of $|\tilde{N}_i|^2$ can be replaced by the pattern average: one has

$$\frac{1}{N}\sum_i |\tilde{N}_i|^2 = \langle |\tilde{N}_i|^2 \rangle_\xi = \alpha |\lambda|^2 q. \tag{31}$$

To represent the noise as $\tilde{N}_i = \sqrt{\alpha r}(x_i + i y_i)/2$, where x_i and y_i are real and obey a normal Gaussian, we define

$$r = 2|\lambda|^2 q = \frac{2q}{|1-u|^2}. \tag{32}$$

Summarizing Eqs. (4), (19), (22), (23), (28), (29), (30), and (32), we have a set of macroscopic order parameter equations

$$s = f[\omega, \tilde{h} + \gamma^{\text{TOTAL}} s, (\tilde{h} + \gamma^{\text{TOTAL}} s)^*], \tag{33}$$

$$\tilde{h} = m + \frac{\sqrt{\alpha r}}{2}(x + iy), \tag{34}$$

$$\gamma^{\text{TOTAL}} = \gamma^{\text{SCSNA}} + \gamma^{\text{TAP}}, \tag{35}$$

$$m = \langle\langle \tilde{f}(\omega, \tilde{h}, \tilde{h}^*) \rangle\rangle, \tag{36}$$

$$q = \langle\langle |\tilde{f}(\omega, \tilde{h}, \tilde{h}^*)|^2 \rangle\rangle, \tag{37}$$

$$\sqrt{\alpha r} u = \langle\langle (x - iy) \tilde{f}(\omega, \tilde{h}, \tilde{h}^*) \rangle\rangle, \tag{38}$$

$$r = \frac{2q}{(1-u)^2}, \tag{39}$$

$$\gamma^{\text{SCSNA}} = \alpha \frac{u}{1-u}, \tag{40}$$

where $\langle\langle \cdots \rangle\rangle$ represents $\langle\langle \cdots \rangle_\omega \rangle_{\tilde{h}} = (1/2\pi \int g(\omega) \exp[-(x^2 + y^2)/2] \cdots d\omega dx dy$. Detailed derivation of Eqs. (38) and (39) is given in Appendix C.

To discuss the generalized expression for γ^{TAP} for the case with a distribution of natural frequencies, we consider for the moment the case with $\omega_i = 0$ $(i = 1, \ldots, N)$, where Eq. (18) exactly holds. In this case it turns out that $\gamma^{\text{TOTAL}} = 0$ by a rough argument given below. Note that $\gamma^{\text{TOTAL}} = 0$ implies $\tilde{f}(\omega, \tilde{h}, \tilde{h}^*) = f(\omega, h, h^*)$, and that the effective transfer function (15) becomes

$$f(0, r, r) = \frac{\int_0^{2\pi} \cos\phi \exp(\tilde{r}\cos\phi) d\phi}{\int_0^{2\pi} \exp(\tilde{r}\cos\phi) d\phi}. \tag{41}$$

Using Eqs. (9) and (41) and performing the average over the Gaussian distribution with unit variance for Eq. (38), we obtain

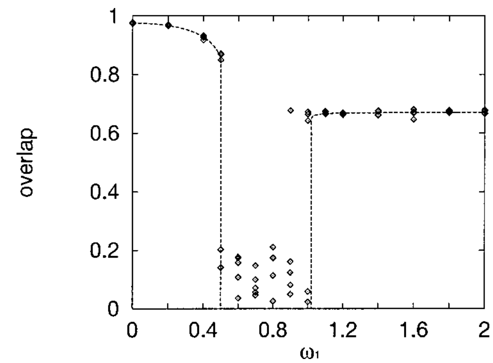

FIG. 3. ω_1 dependence of the overlap obtained from the present analysis is plotted together with the results of numerical simulations with $N = 4000$ for the case with $\alpha = 0.02$, $a = 0.7$, $D = 0$. Since theoretical analysis is based on taking a time average of physical quantities of interest, the results of simulation are displayed in terms of time-averaged quantities $\overline{|m^\mu|} = \overline{|(1/N)\Sigma_i \xi_i^\mu z_i|}$.

$$u = (1-q)/2D. \tag{42}$$

Then, from Eq. (18), we have

$$\gamma^{\text{TAP}} = -\alpha \frac{u}{1-u}, \tag{43}$$

which immediately reconfirms

$$\gamma^{\text{TOTAL}} = \gamma^{\text{SCSNA}} + \gamma^{\text{TAP}} = 0. \tag{44}$$

Now we observe that *in the case with* $\omega_i = 0$ $(i, \ldots, N)$ *the Onsager reaction term* $\gamma^{\text{TAP}} S_i$ *cancels out with the term* $\gamma^{\text{SCSNA}} S_i$ *that emerges as a result of the evaluation of the correlation between the state of oscillators and the stored patterns*. Then *we assume Eq. (43) to hold generally so that* $\gamma^{\text{TOTAL}} = 0$. As will be shown later the results obtained based on this assumption show good agreement with the results of numerical simulations.

IV. BEHAVIORS OF THE NETWORK WITH A DISCRETE NATURAL FREQUENCY DISTRIBUTION

For the sake of simplicity we focus on the behavior of the oscillator network with a discrete natural frequency distribution $g(\omega)$

$$g(\omega) = \frac{1-a}{2}\delta(\omega + \omega_1) + a\delta(\omega) + \frac{1-a}{2}\delta(\omega - \omega_1), \tag{45}$$

where a represents the ratio of the number of oscillators with $\omega_i = 0$ to the total number of oscillators N.

A. Appearance of a window for breakdown of the retrieval states

To roughly sketch the effects of the natural frequency distribution with three frequency components, we investigate the behavior of the overlap with change of ω_1 in the case of $D = 0$. In Fig. 3, we give ω_1 dependence of the overlap calculated from Eqs. (33)–(40) and the result of numerical simulations with $N = 4000$ for the case with $\alpha = 0.02$, a

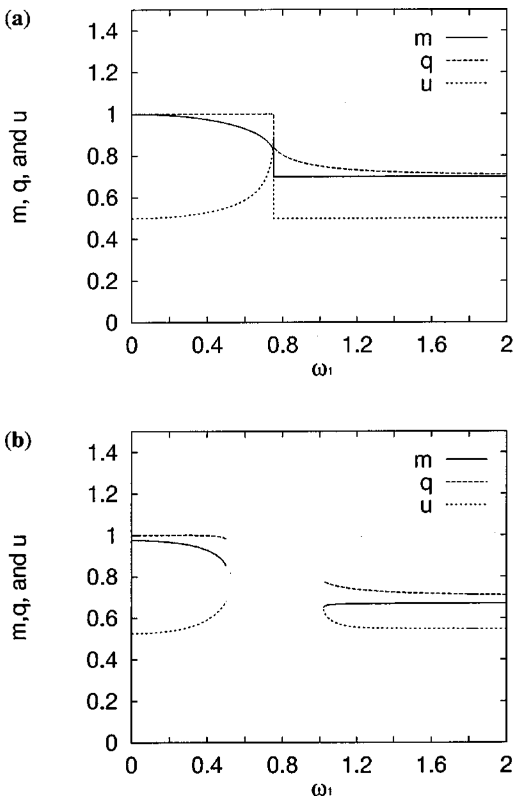

FIG. 4. (a) ω_1-dependences of the order parameters *m*, *q*, and *u* obtained from Eqs. (1), (2), and (3) are displayed in the case with $a=0.7$ and $\alpha=0$. (b) same as (a) for the case with $\alpha=0.02$. The gap separating two types of retrieval states (the large ω_1 regimes and small ω_1 regimes) implies the disappearance of retrieval states.

$=0.7$, and $D=0$. Good agreement between the theory and numerical simulations implies the validity of the present analysis.

As is expected, an entrainment indeed occurs in the case of small $\omega_1(\omega_1 \lesssim 0.5)$, where one has successful retrieval accompanying a large size of overlap. Even in the case of large $\omega_1(1.0 \lesssim \omega_1)$ successful retrieval can be realized with small size of overlap, since natural frequency of *aN* oscillators remains 0.

To give a qualitative explanation for the occurrence of a window for breakdown of the retrieval states, we consider the case with $\alpha=0$ for the moment. We can easily obtain the values of order parameters *m*, *q*, and *u* as functions of ω_1 (see Appendix D) as shown in Fig. 4. We see a phase transition to occur at $\omega_1 \approx 0.7$, and *u* is seen to increase as ω_1 approaches $\omega_1^c : u \to 1$ as $\omega_1 \to \omega_1^c$, while $q=1$ for $\omega_1 < \omega_1^c$ owing to the entrainment.

Even when $\alpha \neq 0$, such an enhancement of *u* around ω_1^c remains unchanged as can be seen in Fig. 4. Noting Eq. (39) we can easily understand the noise variance $\alpha r/2 = \alpha q/(1-u)^2$ may be enhanced accordingly in the interval $0.5 \lesssim \omega_1 \lesssim 1.0$, where retrieval states disappears.

In Fig. 5, we draw the $\omega - \alpha$ phase diagram to show the behavior of the storage capacity as a function of ω_1. The window observed in Fig. 3 turns to arise from the valley of $\alpha_c(\omega_1)$ curves.

B. Effect of white noise

Behaviors of synchronization in the networks of coupled oscillators with white noise differ from those of the deterministic networks with $D=0$. In the absence of white noise ($D=0$), one can in general divide the oscillators with $\omega_i \neq 0$ into two groups of synchronized and desynchronized oscillators according to the criterion of whether the phase velocity of an oscillation vanishes or not. Noting Eq. (36) and the form of effective transfer function with $D=0$ [Eq. (10)] illustrated in Fig. 2, we find that desynchronized oscillators do not contribute to the value of overlap *m*, though they contribute to the value of other order parameters such as *q* and *u*. This is because we are concerned with the time averaged behavior of the local fields, where the time-averaged phase difference between the desynchronized oscillators with natural frequencies ω and $-\omega$ should be π in the absence of white noise [i.e., $f(-\omega,h,h^*) = -f(\omega,h,h^*)$ for $|h|<\omega$]. In the case with white noise ($D\neq 0$), however, the phase of each oscillator with $\omega_i \neq 0$ evolves with a certain non-zero time-averaged phase velocity, since the action of white noise prevents any oscillators from settling into fixed points. Hence, it becomes impossible to distinguish between the synchronized and desynchronized oscillators anymore. Nevertheless a finite value of the overlap is realized because of the existence of the equilibrium probability distribution on $[0,\, 2\pi)$ that is achieved after a long time with $\overline{z} \neq 0$.

In Fig. 6 we display the behavior of the overlap as a function of the noise intensity *D* obtained by the theoretical analysis and numerical simulations. As expected, the size of the overlap decreases as the noise intensity increases until the system undergoes a discontinuous transition at a critical

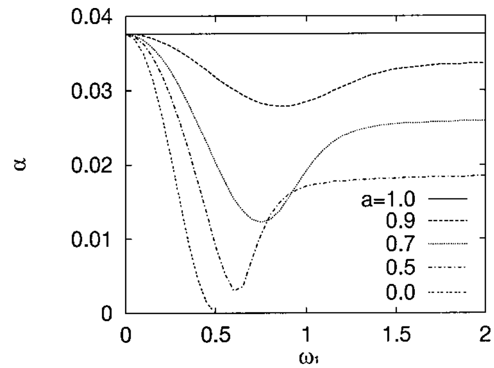

FIG. 5. $\omega_1 - \alpha$ phase diagram representing the behavior of storage capacities for various values of *a* in the case of $D=0$.

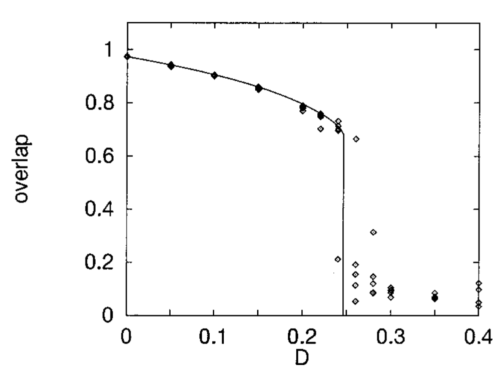

FIG. 6. *D* dependence of the overlap *m* obtained from the present analysis is plotted together with the results of numerical simulations with $N=4000$ in the case with $a=0.7$, $\alpha=0.01$, $\omega_1=0.3$.

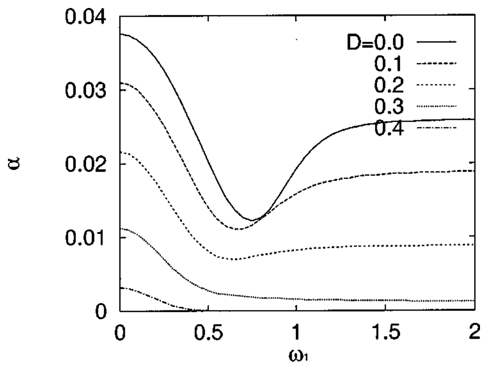

FIG. 7. Same as in Fig. 5 for various values of D in the case of $a=0.7$.

noise intensity D^c, above which a disordered state with $m=0$ is realized. Good agreement between the two results implies the validity of our treatment based on the time averaged local fields together with the assumption that the TAP-like equation also holds in the case with a distribution of natural frequencies.

In Fig. 7, we give $\omega_1-\alpha$ phase-diagram that represents the storage capacity plotted as a function of ω_1 for various values of D. In most of the region of ω_1, the storage capacity α_c decreases as the noise intensity increases. We see that for D smaller than a certain critical D_0, $\alpha_c(\omega_1)$ exhibits non-monotonic behavior with change in ω_1, while for $D>D_0,\alpha_c(\omega_1)$ is monotonically decreasing with ω_1. The occurrence of a window for the breakdown of the retrieval states with fixed α for $D<D_0$ turns out to be attributed to the appearance of a valley caused by the non-monotonic behavior of the $\alpha_c(\omega_1)$ curve as in the case of $D=0$ (Fig. 3).

V. SUMMARY AND DISCUSSIONS

We have investigated properties of an associative memory model of oscillator neural networks based on simple phase oscillators, where the influence of white noise together with a natural frequency distribution is considered in the case of an extensive number of stored patterns. In the presence of white noise every oscillator as well as its local field undergoes fluctuating motions even in the stationary state after a long time. To deal with such a situation we have taken an approach based on the concept of the TAP-like equation. To approximately derive the TAP-like equation for the system without an energy function we have taken the time average for the fluctuating local field of each oscillator neuron to make it constant in time. On the basis of the time-averaged local field we have dealt with the single-body Fokker-Planck equation to obtain the time averaged outputs of the oscillators in the stationary state, from which we have evaluated the effective transfer function. The relation between the time-averaged output and the local fields involving such a transfer function can be viewed as an analogue of the naive TAP-like equation without considering the so called Onsager reaction term. We have supposed the proper form of TAP-like equation to be given by appropriately adjusting the Onsager reaction term such that setting $\omega_i=0$ $(i=1,\ldots,N)$ naturally leads to the legitimate TAP equation, which we have obtained from the evaluation of the Gibbs free energy. Applying the SCSNA to this TAP-like equation, we have obtained the macroscopic order parameter equations, based on which the properties of associative memory of the network have been studied.

Assuming a discrete symmetric natural frequency distribution with three frequency components for the sake of simplicity, we have presented the phase diagram showing the behavior of storage capacity as a function of the parameter ω_1 representing the width of the natural frequency distribution. In the case of $D=0$ the storage capacity α_c has been found to exhibit non-monotonic behavior as ω_1 is varied and to attain a minimum at a certain ω_1. As a result of the occurrence of the valley in the $\omega_1-\alpha_c$ curve the breakdown of the retrieval state with fixed α occurs for intermediate values of ω_1. When noise is present such a behavior has been found to be somewhat relaxed, and only for small values of the noise intensity D the phenomenon of the breakdown of the retrieval state can be observed. Our analytical result has shown excellent agreement with the results of numerical simulations.

Our results show that associative memory based on temporal coding can be realized in the network of simple phase oscillators even in the presence of not only a distribution of natural frequencies but also external white noise. The result that temporal coding is robust against the existence of environmental noise is remarkable. Memory retrieval occurs in such a way that the oscillators undergo synchronized motions with the phase difference $\phi_i-\phi_j$ between any two of the oscillators i and j setting into, for long times, the difference $\theta_i^\mu-\theta_j^\mu$ of the memory pattern μ. In our model θ_i^μ is chosen from uniformly distributed random numbers on [0, 2π). The resultant behavior, however, is qualitatively the same as that for our previous work ($D=0$) [44] on the special case where $\theta_i^\mu=0$ or π and hence J_{ij} is given by the well-known form $J_{ij}=(1/N)\Sigma_\mu\xi_i^\mu\xi_j^\mu$ with $\xi_i^\mu=\pm1$. A characteristic feature of memory retrieval accompanying synchronization is that, in contrast to networks with fixed point type attractors, each neuron exhibits oscillations in the local field or the membrane potential that are easily detected by other neurons in a certain network to determine whether memory retrieval is successful or not. Also worth noting is the appearance of two types of retrieval states with respect to the degree of synchronization: a high degree of synchronization that occurs for small ω_1 with overlap m large and a low degree of synchronization that occurs for large ω_1 with m small.

The fundamental assumption we have used in the present study is the existence of the TAP-like equations (16) and (17) for our system together with the expression of γ^{TAP} [Eq. (43)]. In the case of $\omega_i=0$ $(i=1,\ldots,N)$ there occurs no problem because the genuine TAP equation exists as has been shown. In this case, we have found that γ^{TAP} [Eq. (18)] is exactly canceled out by γ^{SCSNA}, as in the case of the network of AGS, to yield $\gamma^{\mathrm{TOTAL}}=0$ as well as the order parameter equations that recover the ones by Cook [42], who analyzed Q-state spin model including the case with $Q\to\infty$ for arbitrary temperatures by means of the replica symmetric approximation.

In the case with distributed natural frequencies, to estimate the form of the effective transfer function of the TAP-like equation, we have replaced the time-dependent local fields h_i in Eq. (6) by their time-averaged ones, and assumed that the Onsager reaction term of the form: $\gamma^{\mathrm{TAP}} s_i = -\alpha u/(1-u) s_i$ appears in the effective local field as a result of fluctuation of local fields. This form of the Onsager reaction term yields $\gamma^{\mathrm{TOTAL}}=0$, which leads to $\tilde{f}(\omega,h,h^*) = f(\omega,h,h^*)$: The effective transfer function we have derived does not depend on the value of γ^{SCSNA}. This is not the case in more general stochastic networks. A more systematic treatment of the γ^{SCSNA} term in stochastic networks, which is found to justify the present approach for our networks, will be studied elsewhere.

Some special cases of the present model have also been investigated by several authors other than Cook. Arenas *et al.* [41] have investigated the case with $\alpha=0$, where the natural frequency distribution is assumed to obey a Gaussian distribution. The result of this case can also be recovered by the present analysis.

To our knowledge the case with distributed natural frequencies and $\alpha>0$ was first studied by Park *et al.* [52] for different synaptic couplings by means of replica calculations based on the energy that is defined so as to satisfy $\dot{\phi}_i = -dH_{\mathrm{Park}}/d\phi_i + \eta_i$. In our case, the H_{Park} takes the form $H_{\mathrm{Park}} = -\frac{1}{2}\sum_{i\neq j} J_{ij}\cos(\phi_i-\phi_j-\beta_{ij}) - \sum_i \omega_i \phi_i$. However, this energy does not make any sense because the equilibrium distribution $\exp(-H_{\mathrm{Park}}/D)$ does not satisfy the periodical boundary condition $P(\{\phi_i\}) = P(\{\phi_i+2\pi\})$.

Aonishi *et al.* [53] studied the case with $D=0$ and a Gaussian distribution for natural frequencies by considering that $q=1$ holds in the set of SCSNA equations based on a different scheme from ours even in the presence of the group of the desynchronized oscillators.

Yamana *et al.* also studied the deterministic oscillator network ($D=0$) with a discrete distribution of natural frequencies that stores binary patterns by making an approximation that the motions of the group of desynchronized oscillators do not exert an influence on the behavior of the synchronized oscillators. Discarding the effect of desynchronized oscillators corresponds to considering the transfer function that takes the value zero inside the circle with radius ω (see Fig. 2). For a wide class of natural frequency distributions this scheme seems to work to a good approximation in the case with $D=0$, because the contribution of the desynchronized oscillators to such order parameters as m, q, and u is small. It is noted, however, that, in the case of $D\neq 0$, the phase of every oscillator with $\omega_i \neq 0$ evolves with a certain non-zero time-averaged velocity and hence one cannot distinguish between synchronized and desynchronized oscillators. Accordingly for stochastic networks with $D\neq 0$ methods based on neglecting the effect of the desynchronized oscillators will not make sense and one has to deal with all of the oscillators equally as in the present analysis.

Finally, we briefly discuss the relevance of our results to biologically related models of associative memory. Biologically relevant models [31,32,54–57] should be based on such spiking neurons as the Hodgkin-Huxley type [54] and integrate-and-fire type neurons [31,32,56,57]. A simple integrate-and-fire neuron that is defined by one-dimensional linear equation except for firing event can be described in terms of phase that is obtained by properly scaling the one-dimensional output variable. Synaptic couplings implemented into spiking neural networks are often assumed to incorporate the so-called alpha function [56] or its variant represented by the dynamics of a certain gating variables [54,55]. So, major differences between the simple phase oscillator model we have dealt with on the basis of the diffusive couplings among the oscillators and the spiking model will be the form of the synaptic couplings together with symmetry of an individual oscillator with respect to rotation of the phase variable. While the present model is assumed to take a sinusoidal phase interaction for simplicity, a spiking model with a synaptic interaction based on the alpha function takes the form of pulse like couplings [31,32,56,57], which will lead to considering higher harmonics in the phase interaction.

A spiking neural network model of associative memory we previously studied using FitzHugh-Nagumo neurons exhibits a nearly comparable size of the storage capacity to that of the standard analog network with the transfer function $F(h) = [\mathrm{sgn}(h)+1]/2$ that is larger than the storage capacity of the present model [22]. It will then be of interest to observe the outcome of introducing higher harmonics in the phase interaction of the simple phase oscillator model. We expect the storage capacity of the phase oscillator network to increase when the higher harmonics is taken into account. Such an analysis is now under way.

The problem of investigating properties of neurons synchronizing the envelope of a burst of spikes is also of interest, but is beyond the scope of the present paper, which aims at studying the effects of such heterogeneities as a natural frequency distribution and external noise on the robustness of temporal coding in the oscillator network of associative memory. We consider that taking not only phase but also amplitude as variables for oscillatory neurons will provide a solvable model suitable for studying the case with such synchronization in networks of bursting neurons, which is also under way.

ACKNOWLEDGMENT

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APPENDIX A: DERIVATION OF THE TAP EQUATIONS (16) AND (17) IN THE CASE WITH $\omega_i=0$ $(i=1,\ldots,N)$

To obtain the TAP equation for the present model with the energy function (2) we follow the method of Plefka [50] and Nakanishi [51] used for the SK model and neural networks of Ising spins.

The Hamiltonian (2) with a complex-valued external field $R_i + iI_i$ included reads

$$\tilde{H} = aH - \sum_i (R_i\cos\phi_i + I_i\sin\phi_i)$$

$$= -\frac{a}{2}\sum_{i\neq j} c_{ij}^* z_i z_j^* - \sum_i (R_i\cos\phi_i + I_i\sin\phi_i), \quad \text{(A1)}$$

where a is introduced for the analysis below. Applying Legendre transformation to the free energy corresponding to the Hamiltonian $\tilde{H}$, one has

$$G(a,\{s_i\}) = -\beta^{-1}\ln \mathrm{Tr}\exp(-\beta\tilde{H}) + \sum_i (R_i x_i + I_i y_i), \tag{A2}$$

where $\beta = 1/D$ and $s_i = x_i + i y_i = \langle\cos\phi_i\rangle_a + i\langle\sin\phi_i\rangle_a = \langle z_i\rangle_a$. $\langle\cdots\rangle_a$ denotes expectation with respect to the Hamiltonian $\tilde{H}$.

We perform a Taylor expansion with respect to a

$$G(a,\{s_i\}) = \sum_{n=0} \frac{G^{(n)}}{n!} a^n, \tag{A3}$$

where $G^{(n)} = \partial^n G/\partial a^n|_{a=0}$. Noting $B_i + iI_i = \partial G/\partial x_i + i\partial G/\partial y_i$, we rewrite Eq. (A2) in the form

$$\begin{aligned}
G(a,\{s_i\}) &= -\beta^{-1}\ln \mathrm{Tr}\exp\left\{-a\beta H + \beta\sum_{n=0}\sum_i \frac{a^n}{n!}\left(\frac{\partial G^{(n)}}{\partial x_i}\cos\phi_i + \frac{\partial G^{(n)}}{\partial y_i}\sin\phi_i\right)\right\} + \sum_{n=0}\sum_i \frac{a^n}{n!}\left(\frac{\partial G^{(n)}}{\partial x_i}x_i + \frac{\partial G^{(n)}}{\partial y_i}y_i\right)\\
&= -\beta^{-1}\ln Z + \sum_i \left(\frac{\partial G^{(0)}}{\partial x_i}x_i + \frac{\partial G^{(0)}}{\partial y_i}y_i\right) - \beta^{-1}\ln\frac{1}{Z}\mathrm{Tr}\exp\left\{\beta\sum_i\left(\frac{\partial G^{(0)}}{\partial x_i}\cos\phi_i + \frac{\partial G^{(0)}}{\partial y_i}\sin\phi_i\right)\right\}\\
&\quad\times\exp\left\{-a\beta H + \beta\sum_{n=1}\sum_i \frac{a^n}{n!}\left(\frac{\partial G^{(n)}}{\partial x_i}\cos\phi_i + \frac{\partial G^{(n)}}{\partial y_i}\sin\phi_i\right)\right\}\\
&\quad+\sum_{n=1}\sum_i \frac{a^n}{n!}\left(\frac{\partial G^{(n)}}{\partial x_i}x_i + \frac{\partial G^{(n)}}{\partial y_i}y_i\right)\\
&= G^{(0)} - \beta^{-1}\ln\left\langle \exp\left\{-a\beta H + \beta\sum_{n=1}\sum_i \frac{a^n}{n!}\left(\frac{\partial G^{(n)}}{\partial x_i}\cos\phi_i + \frac{\partial G^{(n)}}{\partial y_i}\sin\phi_i\right)\right\}\right\rangle_0\\
&\quad+\sum_{n=1}\sum_i \frac{a^n}{n!}\left(\frac{\partial G^{(n)}}{\partial x_i}x_i + \frac{\partial G^{(n)}}{\partial y_i}y_i\right)\\
&= G^{(0)} - \beta^{-1}\ln\left\langle \exp\left\{-a\beta H + \beta\sum_{n=1}\frac{a^n}{n!}A_n\right\}\right\rangle_0
\end{aligned} \tag{A4}$$

with

$$\begin{aligned}
A_n = \frac{1}{2}\sum_i \{&(\partial G^{(n)}/\partial x_i + i\partial G^{(n)}/\partial y_i)(z_i - s_i)^*\\
&+(\partial G^{(n)}/\partial x_i + i > \partial^{(n)}/\partial y_i)^*(z_i - s_i)\},
\end{aligned} \tag{A5}$$

where $Z = \mathrm{Tr}\exp\{\beta\Sigma_i[(\partial G^{(0)}/\partial x_i)\cos\phi_i + (\partial G^{(0)}/\partial y_i)\sin\phi_i]\}$ and $\langle\cdots\rangle_0$ denotes expectation with respect to the Hamiltonian $\tilde{H}$ with $a=0$.

Noting $s_i = \langle z_i\rangle_a = \langle z_i\rangle_0$, from Eq. (A4), it follows

$$G^{(1)} = \langle H\rangle_0 = -\frac{1}{2}\sum_{i\neq j} c_{ij}^* s_i s_j^*. \tag{A6}$$

Then, from this equation and Eq. (A4), one has

$$\begin{aligned}
G(a,\{s_i\}) &= G^{(0)} + a\langle H\rangle_0 - \beta^{-1}\ln\\
&\quad\times\left\langle \exp\left\{a\beta B + \beta\sum_{n=2}\frac{a^n}{n!}A_n\right\}\right\rangle_0,
\end{aligned} \tag{A7}$$

where

$$B = \frac{1}{2}\sum_{i\neq j} c_{ij}^*(z_i - s_i)(z_j - s_j)^*. \tag{A8}$$

Evaluating $G(a,\{s_i\})$ by expanding this equation upto third order in a yields

$$\begin{aligned}
G(a,\{s_i\}) &= G^{(0)} + \langle H\rangle_0 a - \frac{\beta}{2}\langle B^2\rangle_0 a^2\\
&\quad-\frac{\beta^2}{6}\langle B^3\rangle_0 a^3 + \mathcal{O}(a^4),
\end{aligned} \tag{A9}$$

where it is noted that $\langle B\rangle_0 = \langle A_n\rangle_0 = \langle BA_n\rangle_0 = 0$ for every integer $n \geqslant 1$. Then, noting $\langle z_i - s_i\rangle_0 = \langle (z_i - s_i)^*\rangle_0 = 0$, we have

$$G(a,\{s_i\})=G^{(0)}+\left[-\frac{1}{2}\sum_{i\neq j}c_{ij}^*s_is_j^*\right]a+\left[-\frac{\beta}{16}\sum_{i\neq j}\{E_i(2,0)E_j(0,2)c_{ij}^{*2}+E_i(0,2)E_j(2,0)c_{ji}^{*2}+2E_i(1,1)E_j(1,1)c_{ij}^*c_{ji}^*\}\right]a^2$$
$$+\Big[-\frac{\beta^2}{96}\sum_{i\neq j}\{E_i(3,0)E_j(0,3)c_{ij}^{*3}+E_i(0,3)E_j(3,0)c_{ji}^{*3}+3E_i(2,1)E_j(1,2)c_{ij}^{*2}c_{ji}^*+3E_i(1,2)E_j(2,1)c_{ij}^*c_{ji}^{*2}\}$$
$$-\frac{\beta^2}{48}\sum_{(ijk)}\{E_i(2,0)E_j(1,1)E_k(0,2)c_{ij}^*c_{ik}^*c_{jk}^*+E_i(1,1)E_j(2,0)E_k(0,2)c_{ik}^*c_{ji}^*c_{jk}^*+E_i(1,1)E_j(1,1)E_k(1,1)c_{ij}^*c_{jk}^*c_{ki}^*$$
$$+E_i(0,2)E_j(2,0)E_k(1,1)c_{ji}^*c_{jk}^*c_{ki}^*+E_i(2,0)E_j(0,2)E_k(1,1)c_{ij}^*c_{ik}^*c_{kj}^*+E_i(1,1)E_j(1,1)E_k(1,1)c_{ik}^*c_{ji}^*c_{kj}^*$$
$$+E_i(1,1)E_j(0,2)E_k(2,0)c_{ij}^*c_{ki}^*c_{kj}^*+E_i(0,2)E_j(1,1)E_k(2,0)c_{ji}^*c_{ki}^*c_{kj}^*\}\Big]a^3+\mathcal{O}(a^4), \tag{A10}$$

where $E_i(n,m)=\langle(z_i-s_i)^n(z_i^*-s_i^*)^m\rangle_0$, and (ijk) denotes all combination to be taken so that either two of the indexes do not coincide [note that (ij) implies $i\neq j$]. Then substituting Eq. (A3) into Eq. (A10) yields, in the limit $N\to\infty$,

$$G(a,\{s_i\})=G^{(0)}-\frac{a}{2}\sum_{i\neq j}c_{ij}^*s_is_j^*-\frac{\alpha N\beta(1-q)^2}{8}a^2$$
$$-\frac{\alpha N\beta^2(1-q)^3}{24}a^3+\mathcal{O}(a^4), \tag{A11}$$

where $q=(1/N)\Sigma_i|s_i|^2$. Note that all the relevant terms higher than the term of first order in a under the limit $N\to\infty$ comes from the following in Eq. (10)

$$-\frac{a^n\beta^{n-1}}{2^nn}\sum_{(i_1i_2\ldots i_n)}E_{i_1}(1,1)E_{i_2}(1,1)\cdots E_{i_n}(1,1)$$
$$\times c_{i_1i_2}^*c_{i_2i_3}^*\cdots c_{i_ni_1}^*. \tag{A12}$$

Since every higher order term than the first order one contains $-a^n\beta^{n-1}\langle B^n\rangle_0/n!$, one may expect that it yields terms of the form of Eq. (12). Summarizing those terms we will have

$$G(a,\{s_i\})=G^{(0)}-\frac{a}{2}\sum_{i\neq j}c_{ij}^*s_is_j^*-\alpha N\sum_{n=2}\frac{\beta^{n-1}}{2^nn}(1-q)^na^n. \tag{A13}$$

Then, noting $\partial q/\partial x_i+i\partial q/\partial y_i=2s_i/N$, we obtain

$$R_i+iI_i=\frac{\partial G}{\partial x_i}+i\frac{\partial G}{\partial y_i}=\frac{\partial G^{(0)}}{\partial x_i}+i\frac{\partial G^{(0)}}{\partial y_i}$$
$$-a\sum_{j\neq i}c_{ij}s_j-\gamma^{\mathrm{TAP}}s_i, \tag{A14}$$

where $\gamma^{\mathrm{TAP}}=-a\alpha\{a\beta(1-q)/2\}/\{1-a\beta(1-q)/2\}$.

In the case of $a=0,\tilde{H}$ becomes

$$\tilde{H}=-\sum_i\{(\partial G^{(0)}/\partial x_i)\cos\phi_i+(\partial G^{(0)}/\partial y_i)\sin\phi_i\}. \tag{A15}$$

Thus, we have

$$s_i=\langle\cos\phi_i\rangle_0+i\langle\sin\phi_i\rangle_0$$
$$=\frac{I_1[\beta\sqrt{(\partial G^{(0)}/\partial x_i)^2+(\partial G^{(0)}/\partial y_i)^2}]}{I_0[\beta\sqrt{(\partial G^{(0)}/\partial x_i)^2+(\partial G^{(0)}/\partial y_i)^2}]}$$
$$\times\frac{\partial G^{(0)}/\partial x_i+i\partial G^{(0)}/\partial y_i}{\sqrt{(\partial G^{(0)}/\partial x_i)^2+(\partial G^{(0)}/\partial y_i)^2}}$$
$$=f[0,\partial G^{(0)}/\partial x_i+i\partial G^{(0)}/\partial y_i,(\partial G^{(0)}/\partial x_i+i\partial G^{(0)}/\partial y_i)^*], \tag{A16}$$

where $I_k(r)=(1/\sqrt{2\pi})\int_0^{2\pi}\exp(r\cos\varphi)\cos k\varphi\,d\varphi$, and $f(0,h,h^*)$ is just the effective transfer function we introduced in Eqs. (9), (14), and (15).

Considering the case with $a=1$, from Eqs. (14) and (16) we finally obtain the TAP equation:

$$s_i=f(0,h_i^{\mathrm{TAP}},h_i^{\mathrm{TAP}*}), \tag{A17}$$

$$h_i^{\mathrm{TAP}}=\sum_{j\neq i}c_{ij}s_j+\gamma^{\mathrm{TAP}}s_i+R_i+iI_i, \tag{A18}$$

$$\gamma^{\mathrm{TAP}}=-\alpha\frac{\beta(1-q)/2}{1-\beta(1-q)/2}. \tag{A19}$$

APPENDIX B: DERIVATION OF EQ. (27)

To derive Eq. (27), we substitute Eq. (20) into Eq. (22) to obtain

$$h_i=m^1+\frac{1}{N}\sum_{\mu>1}\sum_{j\neq i}\xi_i^\mu\xi_j^{\mu*}s_j^\mu+\gamma^{\mathrm{TAP}}s_i$$
$$+\frac{\lambda}{N^2}\sum_{\mu>1}\sum_{j\neq i}\sum_{k\neq j}\xi_i^\mu\xi_j^{\mu*}\xi_j^\mu\frac{\partial\tilde{f}}{\partial\tilde{h}}\bigg|_{(\tilde{h}_j^\mu,\tilde{h}_j^{\mu*})}\xi_k^{\mu*}s_k^\mu$$
$$+\frac{\lambda^*}{N^2}\sum_{\mu>1}\sum_{j\neq i}\sum_{k\neq j}\xi_i^\mu\xi_j^{\mu*}\xi_j^{\mu*}\frac{\partial\tilde{f}}{\partial\tilde{h}^*}\bigg|_{(\tilde{h}_j^\mu,\tilde{h}_j^{\mu*})}\xi_k^\mu s_k^{\mu*}. \tag{B1}$$

Utilizing the relations $\xi_j^\mu \xi_j^{\mu *}=1$, $(1/N)\Sigma_i \xi_i^\mu=0$, and so on, the fourth term of Eq. (B1) becomes, in the limit $N\to\infty$,

$$\frac{\lambda}{N^2}\sum_{\mu>1}\sum_{j\neq i}\sum_{k\neq j}\xi_i^\mu \xi_j^{\mu *}\xi_j^\mu \frac{\partial \tilde{f}}{\partial \tilde{h}}\bigg|_{(\tilde{h}_j^\mu,\tilde{h}_j^{\mu *})}\xi_k^{\mu *}s_k^\mu$$

$$=\frac{\lambda}{N}\sum_{\mu>1}\sum_{k}\xi_i^\mu \xi_k^{\mu *}s_k^\mu \frac{1}{N}\sum_{j\neq 1}\frac{\partial \tilde{f}}{\partial \tilde{h}}\bigg|_{(\tilde{h}_j^\mu,\tilde{h}_j^{\mu *})}$$

$$-\frac{\lambda}{N}\sum_{\mu>1}\xi_i^\mu \frac{1}{N}\sum_{j\neq i}\frac{\partial \tilde{f}}{\partial \tilde{h}}\bigg|_{(\tilde{h}_j^\mu,\tilde{h}_j^{\mu *})}\xi_j^{\mu *}s_j^\mu$$

$$=\frac{u\lambda}{N}\sum_{\mu>1}\sum_{k\neq i}\xi_i^\mu \xi_k^{\mu *}s_k^\mu+\alpha u\lambda s_i. \tag{B2}$$

Following the almost same scheme the fifth term of the right hand side of Eq. (B1) is shown to vanish in the limit $N\to\infty$. Substituting Eq. (22) into the left-hand side of Eq. (B1) we obtain Eq. (27).

APPENDIX C: DERIVATION OF EQS. (38) AND (39)

The Eq. (38) is straightforwardly derived from the definition of u by noting $\langle\langle \partial\tilde{f}/\partial\tilde{h}\rangle\rangle=\langle\langle 1/2[\partial\tilde{f}/\partial\{\mathrm{Re}(\tilde{h})\}-i\partial\tilde{f}/\partial\{\mathrm{Im}(\tilde{h})\}]\rangle\rangle$ and performing integration by parts. To show Eq. (39) from Eq. (32) it is suffice to prove that u is real.

To show u is real, note the rotationary symmetry structure of the form of transfer function (9) as is illustrated in Fig. 2. Because of this symmetry structure of $f(\omega,\bar{h},\bar{h}^*)$ we also have $\tilde{f}[\omega,re^{i\theta},(re^{i\theta})^*]=e^{i\theta}\tilde{f}(\omega,r,r)$ in the presence of nonzero complex γ^{TOTAL}. One also immediately finds $f(\omega,r,r)=f(-\omega,r,r)^*$ and $\tilde{f}(\omega,r,r)=\tilde{f}(-\omega,r,r)^*$. Then it follows that $\tilde{f}[\omega,re^{i\theta},(\mathrm{r}e^{i\theta})^*]=\tilde{f}[-\omega,(re^{i\theta})^*,re^{i\theta}]^*$ and $(x-iy)\tilde{f}(\omega,\tilde{h},\tilde{h}^*)=\{(x+iy)\tilde{f}(-\omega,\tilde{h}^*,\tilde{h})\}^*$. On the other hand, noting $g(\omega)=g(-\omega)$ and changing the variables for integration, we have, from Eq. (38),

$$\sqrt{\alpha r}u=\langle\langle (x+iy)\tilde{f}(-\omega,\tilde{h}^*,\tilde{h})\rangle\rangle. \tag{C1}$$

Accordingly, we have

$$\sqrt{\alpha r}u=\langle\langle \{(x-iy)\tilde{f}(\omega,\tilde{h},\tilde{h}^*)\}^*\rangle\rangle=\sqrt{\alpha r}u \tag{C2}$$

to conclude that u is real.

APPENDIX D: DERIVATION OF THE MACROSCOPIC ORDER PARAMETER EQUATIONS FOR THE CASE WITH $D=0$ AND $\alpha=0$

In the case with $D=0$ and $\alpha=0$, substituting Eq. (10) into Eq. (21), we have

$$m=\begin{cases} a, & 0<m\leqslant\omega_1 \\ a+(1-a)\dfrac{\sqrt{m^2-\omega_1^2}}{m}, & \omega_1<m. \end{cases} \tag{D1}$$

Using Eq. (10) we also obtain, from Eqs. (37) and (39), q and u as a function of m:

$$q=\begin{cases} a+(1-a)\left(\dfrac{\omega_1-\sqrt{\omega_1^2-m^2}}{m}\right)^2, & 0<m\leqslant\omega_1 \\ 1, & \omega_1<m \end{cases}, \tag{D2}$$

$$u=\begin{cases} \dfrac{a}{2m}, & 0<m\leqslant\omega_1 \\ \dfrac{a}{2m}+\dfrac{1-a}{2\sqrt{m^2-\omega_1^2}}, & \omega_1<m, \end{cases} \tag{D3}$$

where we have noted $u=\langle\langle \mathrm{Re}\{\partial f/\partial h\}\rangle\rangle=\langle\langle \mathrm{Re}\{(e^{-i\theta}/2)\times(\partial f/\partial r-(i/r)\partial f/\partial\theta)\}\rangle\rangle$, that is obtained by representing the local field with the polar coordinate, i.e. $h=re^{i\theta}$.

As ω_1 approaches the point of phase transition from below, u increases as is shown in Fig. 4. At the phase transition point where $m=a+(1-a)\sqrt{m^2-\omega_1^2}/m$ and $(\partial/\partial m)\{a+(1-a)\sqrt{m^2-\omega_1^2}/m\}=1$, one has $u=1$.

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